Lecture 14. Derived Functors & Cohomology

Note Title 11/2/2019

Abelian category $\mathcal{A}$:

- w/ zero object for the product
- given a morphism $\rightsquigarrow$ kernel & cokernel
- Hom admits abelian group structure
  w/ composition bilinear.

....

ex. category of abelian groups, A-modules, sheaves of $\mathcal{O}_X$-modules

complexes in $\mathcal{A}$, ...

$$a^\bullet := \cdots \to a^{i-1} \xrightarrow{d^{i-1}} a^i \xrightarrow{d^i} a^{i+1} \to \cdots \quad \text{w/} \quad d^i \circ d^{i-1} = 0$$

$$h^i(a^\bullet) = \ker(d^i) / \operatorname{Im}(d^{i-1})$$

Definition: • $I \in \mathcal{A}$ is an injective object if $\operatorname{Hom}(-, I)$ exact.

- $\mathcal{A}$ has enough injective objects if every element
  $a \in \mathcal{A}$ admits an injective resolution. i.e.

$$0 \to a \to I^1 \to I^2 \to \cdots$$

s.t • I^i injective object

• The complex is exact

ex. The category of A-modules, sheaves of $\mathcal{O}_X$-modules

sheaves of abelian groups

have enough injective objects.

$\mathcal{A}$, $\mathcal{B}$ abelian category.

$\mathcal{A}$ has enough injective objects

$F : \mathcal{A} \to \mathcal{B}$ left exact, covariant functor

Given $A \in \mathcal{A}$, choose an injective resolution

$$0 \to A \to I^1 \to I^2 \to \cdots$$

Define the right derived functor $R^iF(A) := h^i(F(I^\bullet))$

Theorem: • $R^iF : \mathcal{A} \to \mathcal{B}$ additive functor independent of the choice of injective resolutions

• $R^0F \simeq F$

$0 \to I \to I \to 0$

$\rightsquigarrow R^iF(I) = 0,\ i > 0$ if I injective.

• If $0 \to A' \to A \to A'' \to 0$ exact in $\mathcal{A}$

then

$$\begin{aligned} &\to R^iF(A') \to R^iF(A) \to R^iF(A'') \\ &\xrightarrow{\delta} R^{i+1}F(A') \to R^{i+1}F(A) \to R^{i+1}F(A'') \\ &\xrightarrow{\delta} R^{i+2}F(A') \to \cdots \end{aligned} \qquad i \geq 0$$

•
$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd} R^iF(A'') \arrow[r, "\delta"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & R^{i+1}F(A') \arrow[d] \\ R^iF(B'') \arrow[r, "\delta"'] & R^{i+1}F(B') \end{tikzcd}$$

Definition: X = topological space.

$\Gamma(X, -)$: category of sheaves of abelian group on X $\longrightarrow$ category of abelian groups

$\rightsquigarrow H^i(X,-) := R^i\Gamma(-)$ i-th cohomology

Remark: ① Take the constant sheaf $\rightsquigarrow H^i(X,R)$
ex. $\mathbb{Z}, \mathbb{R}, \mathbb{C}$

cohomology theory
in algebraic topology

② Refine the category of sheaves of abelian groups on X
to : (quasi-coherent) $\mathcal{O}_X$-modules :

the resulting cohomologies are the same.

However, if $(X, \mathcal{O}_X)$ ringed space w/ $A = \Gamma(X, \mathcal{O}_X)$
then $\Gamma(X, \mathcal{F})$ has an A-module structure
for any sheaf of $\mathcal{O}_X$-module $\mathcal{F}$ on X

Thus, $H^i(X, \mathcal{F})$ naturally admits an A-module structure.

X: topological space
$\mathcal{F}$: sheaf of abelian group on X

Definition: $\mathcal{F}$ is <u>flasque</u> if $\mathcal{F}(U) \twoheadrightarrow \mathcal{F}(V)$ for $V \subseteq U$

Lemma 1: $(X, \mathcal{O}_X)$ ringed space, injective $\mathcal{O}_X$-modules are flasque.

pf: $U \underset{\text{open}}{\subseteq} X \rightsquigarrow \mathcal{O}_U := j_!(\mathcal{O}_X|_U)$
extending by zero outside U

$\underset{\text{open sets}}{V \subseteq U} \rightsquigarrow 0 \to \mathcal{O}_V \to \mathcal{O}_U$

$\mathcal{I}$: Injective sheaf of $\mathcal{O}_X$-module

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_U, \mathcal{I}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, \mathcal{I}) \longrightarrow 0 \quad \text{exact}$$

$$\underset{\mathcal{I}(U)}{\parallel} \qquad\qquad \underset{\mathcal{I}(V)}{\parallel}$$

Lemma 2: $\mathcal{F}$: flasque sheaf on ringed space $(X, \mathcal{O}_X)$

then $H^i(X, \mathcal{F}) = 0$, $i > 0$

pf: $\exists\, \mathcal{I}$ injective s.t. $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$ $\Rightarrow \mathcal{G}$ flasque

$\mathcal{F}$ flasque, $\mathcal{I}$ injective $\Downarrow$ flasque

$$H^i(X, \mathcal{I}) \longrightarrow H^i(X, \mathcal{G}) \xrightarrow{\cong} H^{i+1}(X, \mathcal{F}) \longrightarrow H^{i+1}(X, \mathcal{I}), \quad i > 0$$

$\mathcal{I}$ injective: $H^i(X, \mathcal{I}) = 0$, $H^{i+1}(X, \mathcal{I}) = 0$

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{I}) \longrightarrow \Gamma(X, \mathcal{G}) \longrightarrow 0 \quad + \quad H^1(X, \mathcal{I}) = 0$$

$$\Downarrow$$

$H^1(X, \mathcal{F}) = 0$ for any flasque sheaf

Thus, $\underset{H^2(X, \mathcal{F})}{H^1(X, \mathcal{G})} = 0 \xRightarrow[\text{induction}]{} H^i(X, \mathcal{F}) = 0, \ \forall i > 0$

Theorem (Grothendieck)

X: Noetherian topological space of dimension n

$\mathcal{F}$: sheaf of abelian group on X

$\Longrightarrow \quad H^i(X,\mathcal{F}) = 0, \quad \forall\, i > n$

pf: $\underset{\text{closed}}{Y \overset{j}{\hookrightarrow} X}, \quad \mathcal{F}_Y := j_*(\mathcal{F}|_Y)$

$\underset{\text{open}}{U \overset{i}{\hookrightarrow} X}, \quad \mathcal{F}_U := i_!(\mathcal{F}|_U)$

If $U = X \setminus Y$, then $0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Y \to 0$

We will prove by induction on $n = \dim X$ w/ several reductions.

Step 1. (Reduction to X irreducible)

$Y \hookrightarrow X$ an irreducible component of X

$U = X \setminus Y$

$0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_Y \to 0$

$\underline{H^i(Y,\mathcal{F}_Y)} = 0 = \underline{H^i(U,\mathcal{F}_U)}, \; i > n \Longrightarrow H^i(X,\mathcal{F}) = 0, \; i > n$

Y: irreducible

U has # of components less than X

Lemma 3: $\underset{\text{closed}}{Y \overset{j}{\subseteq} X}$, $\mathcal{F}$ sheaf of abelian group on Y

then $H^i(X, j_*\mathcal{F}) = H^i(Y,\mathcal{F})$

Step 2. $\dim X = 0$,

Then the only open subsets of X are empty set & X itself.

Otherwise, X admits proper closed subset $\Rightarrow \dim X > 0$

In this case,

$$\text{category of sheaves of abelian groups on } X \overset{\Gamma}{\simeq} \text{category of abelian groups}$$

$$0 \to \mathcal{F} \to I^1 \to I^2 \to \cdots \Rightarrow H^i(X, \mathcal{F}) = 0, \ i > 0$$

$$H^i(X,\mathcal{F}) \overset{\text{def}}{=} h^i(\Gamma(I^\bullet)) = h^i(I^\bullet) = 0$$

$\because I^\bullet$ exact

Step 3. Now X irreducible of dimension n.

Let $B = \bigcup_{\substack{U \subseteq X \\ \text{open}}} \mathcal{F}(U)$, $A = \{\text{finit subset of } B\}$ directed

$\forall \alpha \in A \rightsquigarrow \mathcal{F}_\alpha =$ subsheaf of $\mathcal{F}$ generated by elements in α.

$$\mathcal{F} = \varinjlim \mathcal{F}_\alpha$$

Lemma 4. $H^i(X, \varinjlim \mathcal{F}_\alpha) = \varinjlim H^i(X, \mathcal{F}_\alpha)$

Thus, it suffices to prove that $H^i(X, \mathcal{F}_\alpha) = 0, \ \forall i > n$

If $\alpha' \subseteq \alpha$, then

$$0 \to \mathcal{F}_{\alpha'} \to \mathcal{F}_\alpha \to \mathcal{G} \to 0$$

$\mathcal{G}$ generated by $|\alpha| - |\alpha'|$ sections

By induction on $|\alpha|$, it suffices to prove the case $\mathcal{F}$ is generated by 1 section over some open set U.

i.e $\quad 0 \to \mathcal{R} \to \mathbb{Z}_U \twoheadrightarrow \mathcal{F} \to 0$

Only need to prove the theorem for $\mathcal{R}$ & $\mathbb{Z}_U$.

Step 4. $U \underset{\text{open}}{\subseteq} X$, $\mathcal{R}$ subsheaf of $\mathbb{Z}_U$

$\forall x \in U$, $\mathcal{R}_x = (d_x) \subseteq \mathbb{Z}$

If $d_x = 0$, $\forall x \in U$, then $\mathcal{R} \equiv 0$

Otherwise, $d_x \neq 0$, $\exists V \underset{\text{open}}{\subseteq} U$ s.t $\mathcal{R}|_V \cong d \cdot \mathbb{Z}_V \subseteq \mathbb{Z}_V$

Thus, $\quad 0 \to \mathbb{Z}_V \to \mathcal{R} \to \underbrace{\mathcal{R}/\mathbb{Z}_V}_{} \to 0$

support on $U \setminus V$

w/ dimension $\leq n-1$ $\quad U$: irreducible

can apply induction hypothesis

Thus, we only need to prove the theorem for $\mathcal{F} = \mathbb{Z}_U$.

Step 5. $U \underset{\text{open}}{\subseteq} X \quad Y = X \setminus U$ closed $\quad \dim Y < \dim X$

$$0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0$$

$H^i(X, \mathbb{Z}_Y) = 0$, $\forall i > n-1$ $\quad$ induction hypothesis

$H^i(X, \mathbb{Z}) = 0, \quad \forall i > 1$

Constant sheaf on irreducible space is flasque

$\Rightarrow H^i(X, \mathbb{Z}_U) = 0, \quad \forall i > n$

(Lemma 3 proof) $\quad 0 \to \mathcal{F} \to \mathcal{J}^\bullet$

flasque resolution of $\mathcal{F}$

$\Rightarrow j_* \mathcal{J}^\bullet$ flasque resolution of $j_* \mathcal{F}$

$\Rightarrow H^i(X, j_* \mathcal{F}) = h^i(\Gamma(X, j_* \mathcal{J}^\bullet)) = h^i(\Gamma(Y, \mathcal{J}^\bullet)) = H^i(Y, \mathcal{F})$

- $\mathcal{F}_p := \varinjlim_{\substack{p \in V \subseteq Y \\ \text{open}}} \mathcal{F}(V) = \varinjlim_{\substack{p \in U \subseteq X \\ \text{open}}} j_* \mathcal{F}(U) =: (j_* \mathcal{F})_{j(p)}$

Thus $j_* \mathcal{J}^\bullet$ is a resolution of $j_* \mathcal{F}$

- $$\begin{array}{ccc} j_* \mathcal{J}^i(U) & \cong & \mathcal{J}^i(j^{-1}(U)) \\ \downarrow & & \downarrow \\ j_* \mathcal{J}^i(V) & \cong & \mathcal{J}^i(j^{-1}(V)) \end{array}$$

$\because \mathcal{J}^i$ flasque $\Rightarrow j_* \mathcal{J}^i$ flasque

$V \subseteq U$